Similarity of Motion in Relation to the Surface Friction of Fluids

T. E. Stanton and J. R. Pannell

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V. *Similarity of Motion in Relation to the Surface Friction of Fluids.*

By T. E. STANTON *and* J. R. PANNELL.

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(*From the National Physical Laboratory.*)

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THE laws of the surface friction of fluids have formed the subject of many important investigations during the last 100 years, among which may be mentioned the work of POISEUILLE, DARCY and OSBORNE REYNOLDS on the friction of water flowing in pipes, that of WILLIAM FROUDE on the resistance of thin plates towed in water, and the corresponding experiments of ZAHM on flat plates in a current of air. Researches in this field have also been carried out by BRIX, STOCKALPER, MALLOCK, COKER, GEBERS, BRIGHTMORE, GRINDLEY and GIBSON, and others.

As a result, the effect on the resistance, of the dimensions of the body over whose surface the fluid moves, and of the velocity of flow, are tolerably well known for the particular fluid and character of motion observed. In the case of the surface friction of water in pipes, the researches of OSBORNE REYNOLDS have demonstrated the existence of similar motions in pipes of different dimensions, but, as far as the authors are aware, no systematic series of experiments appears to have been made for the purpose of establishing a general relation which would be applicable to all fluids and conditions of flow, although the existence of such relationships for different aspects of the problem were predicted as a consequence of the laws of motion by STOKES in 1850,[*] by HELMHOLTZ in 1873,[†] by OSBORNE REYNOLDS in 1882,[‡] by Lord RAYLEIGH[§][||] in 1899 and 1909, and as has been pointed out by Sir GEORGE GREENHILL, were foreshadowed by NEWTON in Proposition 32, Book II., of the 'Principia.'

The object of the present paper is to furnish evidence confirming the existence,

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[*] STOKES, 'Mathematical and Physical Papers,' vol. III., p. 17.

[†] HELMHOLTZ, 'Wissenschaftliche Abhandlungen,' vol. I., p. 158.

[‡] 'Phil. Trans. Roy. Soc.,' 1883, p. 935.

[§] 'Phil. Mag.,' 1899, p. 321.

[||] 'Advisory Committee for Aeronautics, Report,' 1909–10, p. 38.

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under certain conditions, of the similarity in motions of fluids of widely differing viscosities and densities which has been predicted, and further by extending the observations through a range in the velocity of flow which has not hitherto been attempted to investigate the limits of accuracy of the generally accepted formulæ used in calculations of surface friction.

Previous Experimental Investigations.

Apart from the researches on similarity of motion of fluids, which have been in progress in the Aeronautical Department of the National Physical Laboratory during the four last years, the only previous experimental investigation on the subject, as far as the authors are aware, has been that of OSBORNE REYNOLDS, to which a brief reference may be made.

By the introduction of colouring matter into water flowing through glass tubes REYNOLDS showed that the motion was stream-line or lamellar in character at low values of the velocity of flow, and eddying or sinuous at high velocities, and that the change from lamellar motion to eddying motion took place suddenly at a definite value of the velocity (called the critical velocity), the value of which was inversely proportional to the diameter of the tube and directly proportional to the kinematical viscosity of the water.

Expressing this in symbols, if

d = diameter of the pipe,
v_c = the critical velocity,*
μ = the coefficient of viscosity of the water,
ρ = the density of the water,
ν = kinematical viscosity of the water ($=\mu/\rho$),
dp/dx = rate of fall of pressure along the pipe.

REYNOLDS's discovery was that for geometrically similar tubes

$v_c d/\nu$ was constant.

Further, on making a series of observations of the values of dp/dx over as large a range in the velocity of flow as possible in similar pipes of different diameters, REYNOLDS found that for all conditions of flow, stream line or eddying, when the values of vd/ν were identical the corresponding values of $\rho d^3/\mu^2 \,.\, dp/dx$ were identical. It appeared, therefore, that the general law of resistance could be expressed by the equation

$$\frac{\rho d^3}{\mu^2}\cdot\frac{dp}{dx}=f\left(\frac{vd}{\nu}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

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* Throughout the present paper the symbol v is used to denote the mean velocity of flow through the pipe. Where reference is made to the value of the velocity at the axis of the pipe this is denoted by $v_{\max}$.

Comparing this equation with the theoretical value of dp/dx for stream-line motion, *i.e.*,

$$\frac{dp}{dx}=\frac{32\mu v}{d^2},$$

it will be seen that under conditions of stream-line motion

$$f\left(\frac{vd}{\nu}\right)\propto\frac{vd}{\nu}.$$

For velocities above the critical value, and as high as could be carried in his apparatus, REYNOLDS considered that the function on the right-hand side of the equation could be expressed by the relation

$$f\left(\frac{vd}{\nu}\right)\propto\left(\frac{vd}{\nu}\right)^n,$$

where n has a value which is constant for any given pipe, but may vary from 1·75 to 2 according to the roughness of the surface. As will be seen later, this conclusion has not been verified in the present experiments, which show definitely that when the range in speed is considerable the index law fails to represent the results.

It has been shown by Lord RAYLEIGH that by the Principle of Dynamical Similarity the relation expressed by equation (1) is only a particular case of a general law of resistance of bodies immersed in fluids moving relatively to them, under the assumption that this resistance depends only on the linear dimensions of the body and on the velocity, density, and kinematical viscosity of the fluid. This relation may be expressed as

$$\mathrm{R}=\rho v^2\mathrm{F}\left(\frac{vd}{\nu}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where R is the resistance per unit area and F is a function of the one variable vd/ν.

From the foregoing it appears that similarity of motion in fluids at constant values of the variable vd/ν will exist, provided the surfaces relative to which the fluids move are geometrically similar, which similarity, as Lord RAYLEIGH has pointed out, must extend to those irregularities in the surfaces which constitute roughness. In view of the practical value of the ability to apply this principle to the prediction of the resistance of aircraft from experiments on models, experimental investigation of the conditions under which similar motions can be produced under practical conditions becomes of considerable importance, and during the last three years the accuracy of the assumptions made in the derivation of equation (2) has been tested at the National Physical Laboratory under varying conditions. By the use of colouring matter to reveal the eddy systems at the back of similar inclined plates in streams of air and water, photographs of the systems existing in the two fluids when the value of vd/ν was the same for each, have been obtained, and their comparison has revealed a remarkable similarity in the motions.* For the case of surface friction, experiments

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* 'Report of Advisory Committee for Aeronautics,' 1911–12, p. 97.

have been made on pipes whose surfaces were smooth or artificially roughened so that the amount of the friction was more than doubled. By making observations on the same pipe first with air flowing through it and then with water at identical values of vd/ν it was found that the corresponding values of $R/\rho v^2$ were also identical both for the smooth pipe and for the rough one.*

Comparisons have also been made between the resistances of models of dirigible balloons made to different scales, in one case when exposed to a current of air and in the other case when towed in an experimental water tank, and results confirmatory of the theory have been obtained.

The experiments described in the present paper form a continuation of the investigation with reference to surface friction, so as to include the highest range in the velocities of flow, the dimensions of the surfaces, and in the nature of the fluids used, as could be conveniently obtained. It will be realised that in order to vary the conditions to the extent desired, and to measure the resistances with the requisite accuracy, the research was practically limited to observations on the resistance to the flow of fluids in pipes. Adopting this method of working, the velocities of flow which have been obtained have ranged from 30 to 6000 cm. per second in pipes varying from 0·3 to 10·0 cm. in diameter. The fluids used in the majority of the experiments have been air and water. The physical properties of these are so widely different that observations on others are hardly necessary, but as a matter of interest some experiments on thick oils are included.

For very accurate comparison the surfaces of the tubes should have been precisely geometrically similar, as regards roughness, but as this condition could not be fulfilled, the experiments were all made on commercially smooth-drawn brass pipes. From the general agreement of the results of different pipes it does not appear that slight irregularities in this respect have a marked effect on the resistance within the range of diameters here used.

Nature of the Observations in the Experiments.

To set up the state of motion required the experimental pipe was, in the majority of cases, connected to a centrifugal fan or pump, driven by an electric motor and provided with speed-regulating devices by means of which the flow of the fluid could be maintained constant throughout the duration of any particular experiment. The length of "leading in" pipe, of the same diameter as the experimental portion, through which the fluid passed before any observations of its velocity or pressure were made, varied from 90 to 140 diameters, as it was considered that this length was sufficient both to enable any irregularities in the distribution of velocity to die away, or any stream-line motion at the inlet to break up, before the measurements were taken.

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* 'Roy. Soc. Proc.,' A, vol. 85, p. 366.

To measure the velocity of the current, one of two methods was used according to convenience. By one method the total quantity of fluid passing through the pipe in a given time was either weighed directly, or passed through a water-meter or a gas-holder, which had been designed for the purpose of the experiments and carefully calibrated. By the other method the velocity at the axis of the pipe was estimated by measuring the difference of pressure between that in a small Pitot tube facing the current and placed in the axis of the pipe and that in a small hole in the wall of the pipe. As is now known* this pressure difference is accurately $\frac{1}{2}\rho v^2$, and from this relation the speed at the axis was determined. The mean flow was then calculated from the known ratio of the mean speed to the speed at the axis; an investigation into which forms part of the present paper.

To determine the amount of the surface friction two small holes were made in the walls of the experimental portion of the pipe, one at each extremity, at a known distance apart, and connected to a tilting manometer. The length of this portion varied in different pipes from 20 to 50 diameters, according to the resistance to be measured. In this way the fall of pressure along a given length of the pipe was determined, and from the known diameter of the pipe the surface friction per unit area was calculated. The form of tilting manometer used for the estimation of both the surface friction and the axial velocity, is that devised by Dr. A. P. Chattock and has been previously described.† For the purpose of the present paper it is sufficient to state that in this manometer a pressure difference of the order of 0·003 mm. of water can easily be detected, which is well within the limits of sensitivity required in these experiments. As the fall of pressure in these pipes varied from 0·5 to 150,000 mm. of water, other manometers were required for the higher pressures, and for this purpose water or mercury U-tubes were used for the intermediate pressures, and Bourdon pressure gauges for the highest pressures.

The Relation between the Mean Velocity of Flow and the Velocity at the Axis in Tubes of Circular Cross Section.

In the case of the "stream-line" motion of a viscous fluid through a tube, to which the ordinary equations of motions apply, it appears both from theory and direct experiment that the ratio of the mean velocity over the section to the velocity at the axis is 0·5. For the case of eddying flow the value of this ratio has been investigated by Darcy‡ for water and by Threlfall§ for air. It might be inferred from the conditions of similarity of flow discussed above, that the ratio in question would be a function of the ratio vd/ν, and in a paper on the Mechanical Viscosity of Fluids read

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* 'Report of Advisory Committee for Aeronautics,' 1912–13.

† 'Proc. Inst. Civil Engineers,' December, 1903, and 'Engineering,' September, 1913.

‡ 'Comptes Rendus de l'Académie des Sciences,' vol. 38.

§ 'Proc. Inst. Mechanical Engineers,' 1904, p. 280.

before the Society in 1911* the results of experiments on the radial variation of the velocity of air flowing through smooth brass pipes of 4·9 and 7·4 cm. diameter were given, showing that the velocity distribution curves were only identical when the values of the ratio vd/ν were identical. It was also found that the distribution of velocity was only independent of the values of vd/ν when the surfaces of the pipes were so roughened that the resistance varied as the square of the velocity.

On examining the results of DARCY and THRELFALL no very satisfactory evidence of a variation of the ratio with variations of vd/ν was forthcoming. This may be due to the fact that THRELFALL'S observations are all included in a range of vd/ν of from 66,000 to 300,000, and as the variation in the ratio may be assumed to be small, the difference between the results would not be detected very easily, and also to the possibility that DARCY'S observations were made in a region over which the curve showing the variation of the ratio with vd/ν was nearly horizontal.

As an accurate determination of the ratio was essential for many of the proposed friction experiments it was decided to make an extensive series of observations with a view to tracing the variation of the ratio from its value 0·5 below the critical value of the ratio vd/ν to as high a value as possible. As the use of an accurately calibrated gas holder was available it was possible to make the determination for the case of air flow as well as for water flow. The pipes used were 0·7125, 1·255 and 2·855 cm. in diameter, and the method of estimating the velocity at the axis was that previously described. It may be mentioned that in this part of the work comparatively small errors in observation would have obscured the effect to be investigated, as its whole amount in the region of eddying flow is of the order of 5 per cent., so that the greatest care and accuracy was necessary.

The calculated values of the ratio of mean velocity to the velocity at the axis are plotted in fig. 1 as ordinates with the corresponding values of $\log v_{\max}.d/\nu$ as abscissæ. The values of the velocity at the axis have been adopted merely for convenience in the use of the curve to determine rates of discharge from a pipe by means of a single observation at the axis, and experience has shown that in this work a better graphical representation of the results is achieved by using a logarithmic scale for plotting the values of vd/ν than by using a simple one.

The observations were continued for smaller and smaller values of vd/ν until stream-line motion was set up, below which it was not considered necessary to go.

The values of the calculated ratios are tabulated in Table I. It will be seen from the mean curve drawn through the plotted points in fig. 1 that the value of the ratio ranges from 0·5 at the critical value of the speed, *i.e.*, at $vd/\nu = 2500$, to 0·81 at the value $vd/\nu = 70,000$. For the comparison of these results with those obtained by other experimenters the mean curve through the National Physical Laboratory results is reproduced in fig. 2, in which the results of DARCY and THRELFALL are plotted. It will be noticed that the curve passes through two of THRELFALL'S points whereas

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* 'Roy. Soc. Proc.,' A, vol. 85, p. 366.

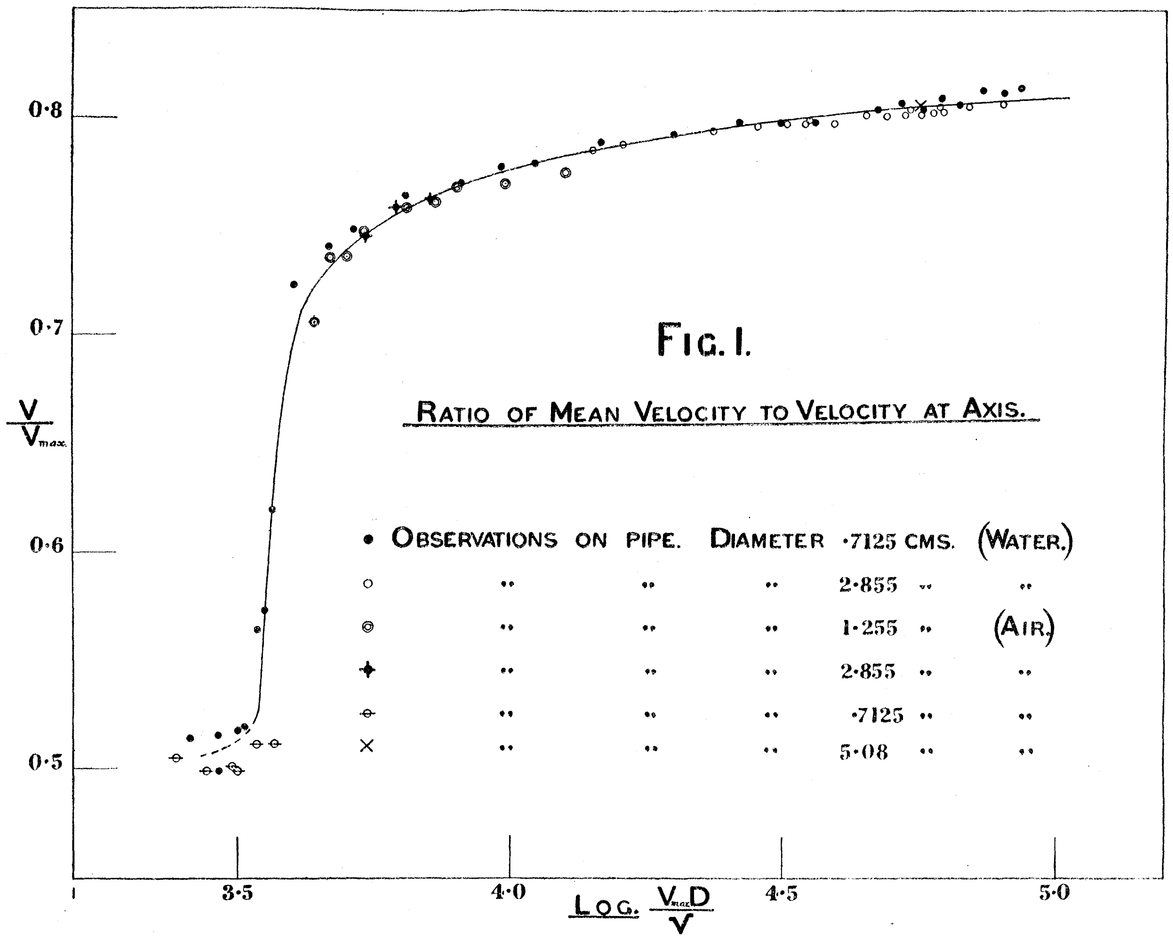

FIG. I.

RATIO OF MEAN VELOCITY TO VELOCITY AT AXIS.

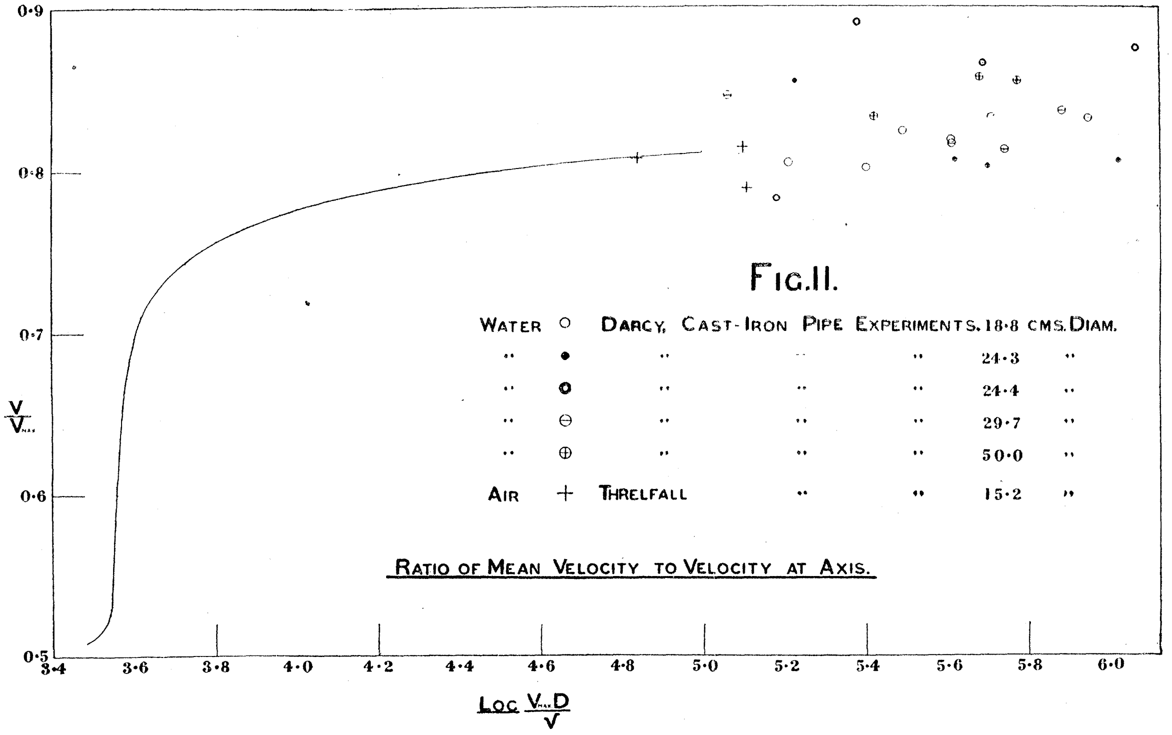

FIG. II.

RATIO OF MEAN VELOCITY TO VELOCITY AT AXIS.

The curve represents the author's mean results.

the third is not in agreement. It is possible that this may be due to the third pipe being rougher in surface than the others, as it has been shown by direct observation of distribution of velocity in rough and smooth pipes at the National Physical Laboratory* that the ratio of mean speed to maximum speed in a rough pipe is less than that in a smooth one.

The interpretation of DARCY'S results is also difficult for the same reason. According to the determinations of the frictional resistances of these cast-iron pipes given by DARCY, their surfaces are all appreciably rougher than those used in the present experiments, so that the ratio in the case of the former should be less than for the latter, but no direct evidence of this is obtained from the plotted results. In fig. 1 is also plotted the ratio obtained by direct integration from the radial distribution of the velocity in a 5·08 cm. pipe, which agrees well with the other results.

The Surface Friction Experiments.

The ratio of the mean velocity to the velocity at the axis having been determined, it was possible to complete the observations of surface friction for those cases in which no other method of estimating the discharge was available.

The scope of the experiments will be seen from the table on p. 207 in which the dimensions of the pipes, and the methods of setting up the flow and estimating the velocity and friction per unit area are stated. The actual values of the mean velocity of flow in centimetres per second and the surface friction in dynes per square centimetre of the surface are given in the tables at the end of the paper together with the calculated values of the ratio vd/ν (Tables II. and III.).

The method of representing the results is that, suggested by Lord RAYLEIGH, of plotting points whose ordinates are the values of $R/\rho v^2$ and abscissæ the corresponding values of vd/ν, with the previously stated modification that the logarithms of vd/ν are taken as the abscissæ instead of the actual values. In this way it has been found possible to include all the results on a reasonable length of diagram, and at the same time to show the region of the change from eddying to stream-line motion on a fairly open scale. This has been done in fig. 3.

The extension of the experiments for values of vd/ν greater than 115,000, which was the maximum attainable with the appliances available at the National Physical Laboratory, has been made possible by the kindness of Mr. Edward B. Ellington, Engineer of the London Hydraulic Power Supply Company, who has allowed the authors to connect their experimental pipes to the high-pressure water main at the Grosvenor Road Pumping Station. In this way it has been found possible to reach velocities of 3200 cm. per second in the 1·255 cm. diameter pipe, 5250 cm. per second in the 0·7125 cm. pipe, and 5600 cm. per second in the 0·361 cm. pipe. The maximum value of vd/ν attained was 430,000.

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* 'Roy. Soc. Proc.,' A, vol. 85, p. 371.

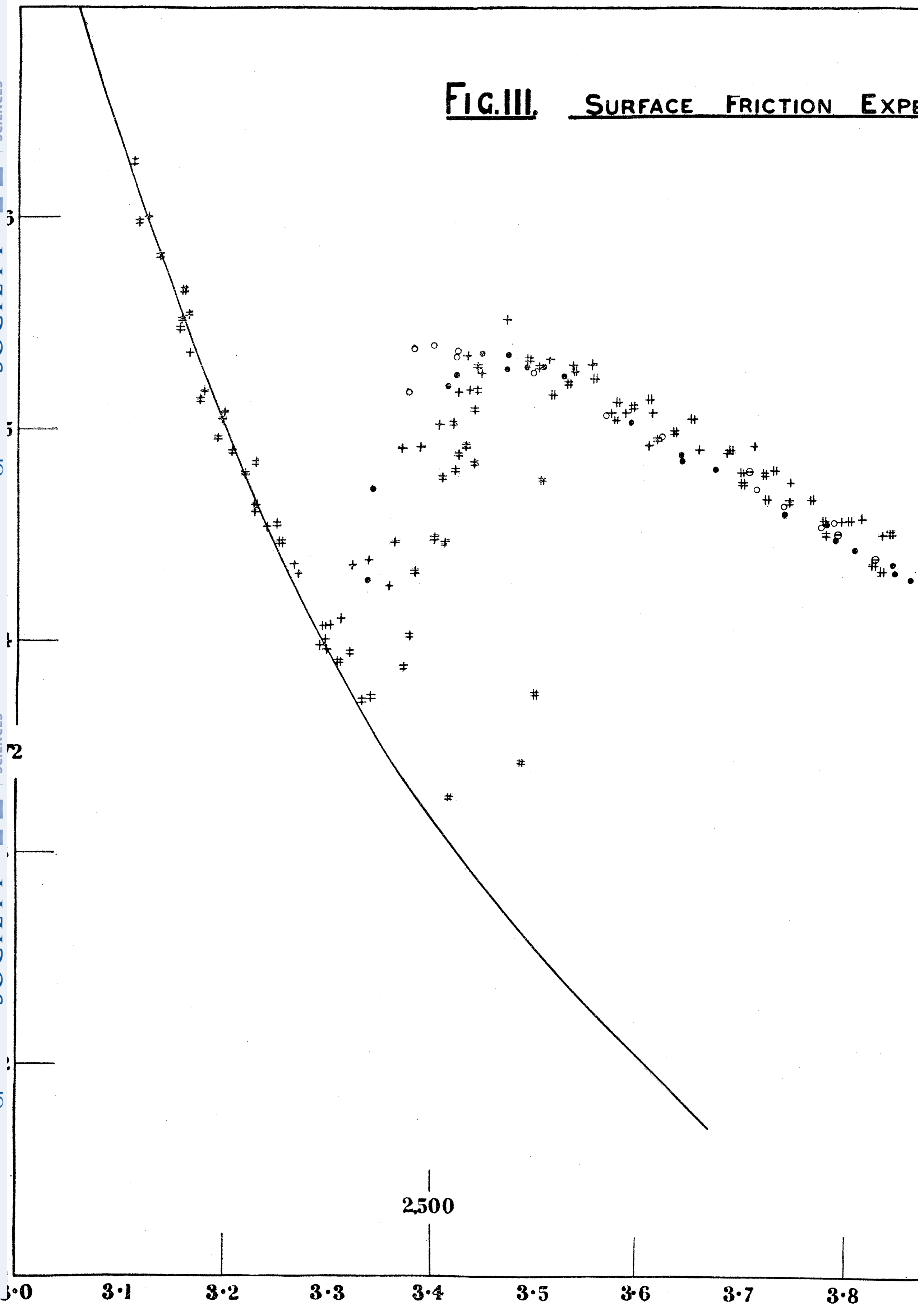
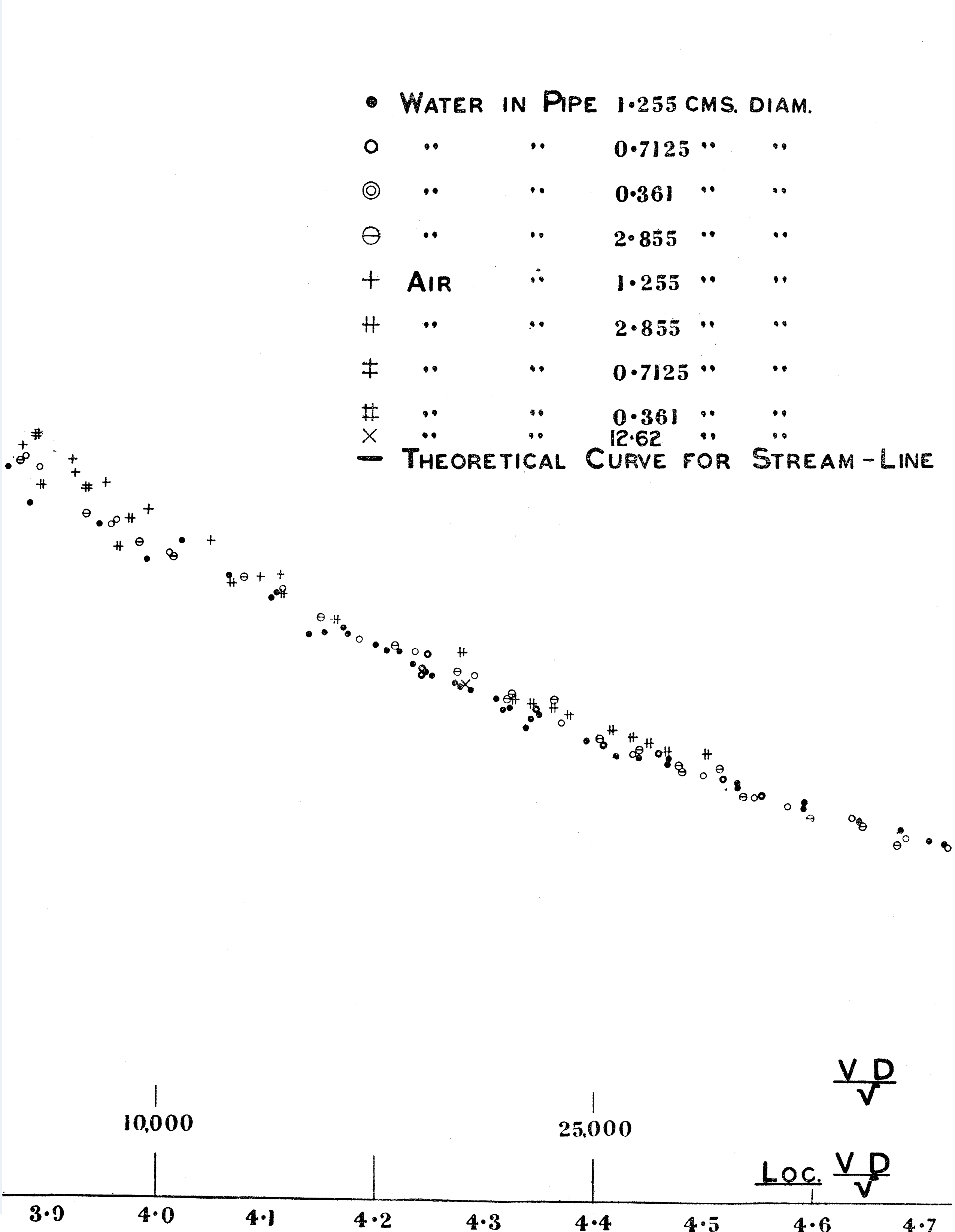


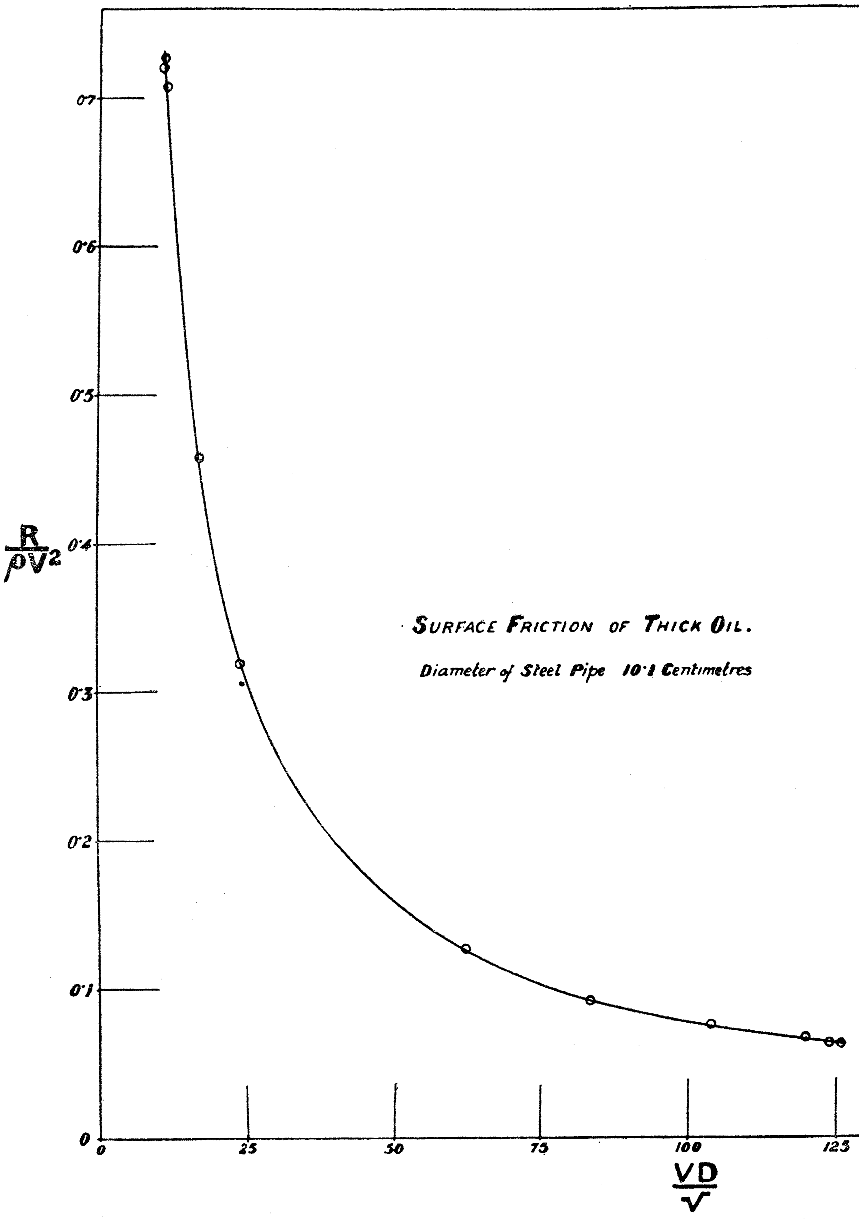
FIG. III. SURFACE FRICTION EXPE

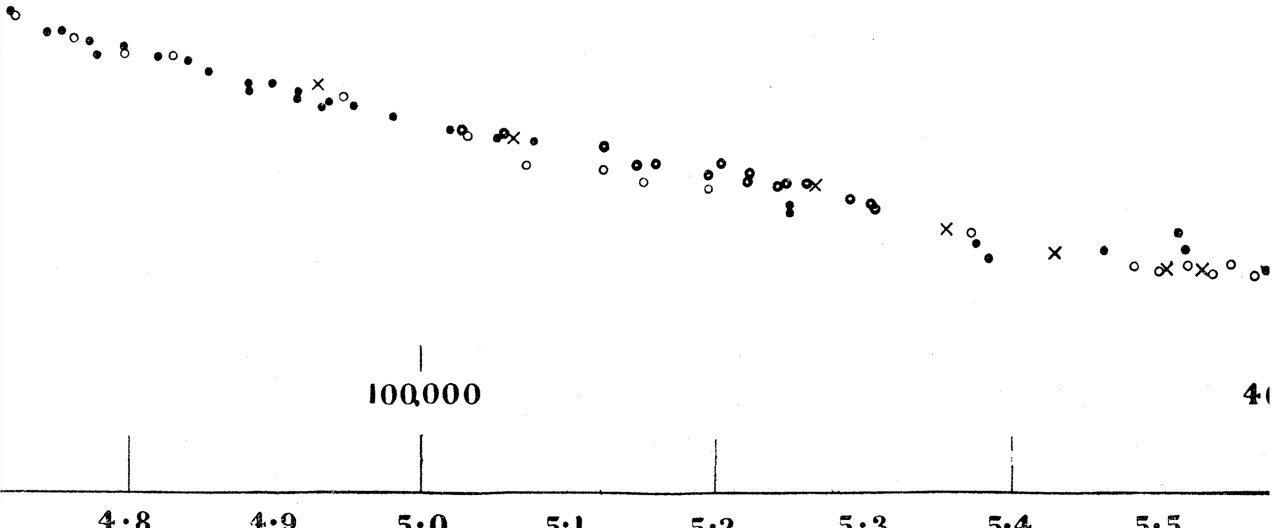
PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

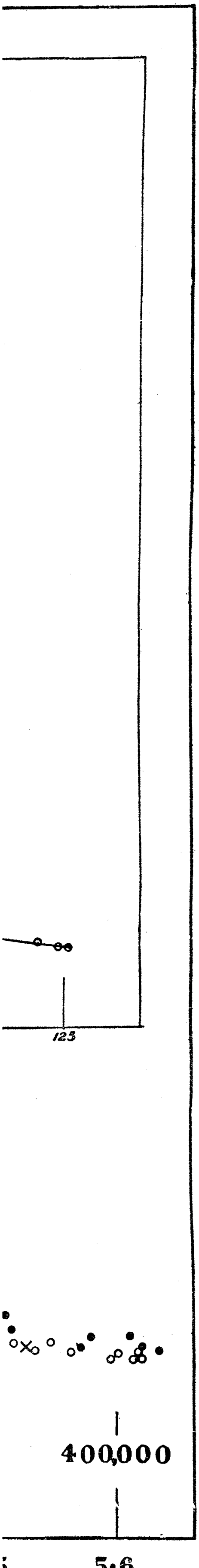
PERIMENTS WITH AIR AND WATER.

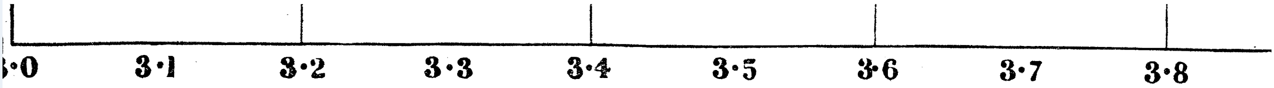


MOTION.


SURFACE FRICTION OF THICK OIL.
Diameter of Steel Pipe 10·1 Centimetres
$\frac{R}{\rho V^2}$
0·7
0·6
0·5
0·4
0·3
0·2
0·1
0
0
25
50
75
100
125
$\frac{VD}{\nu}$


100,000
4
4·7
4·8
4·9
5·0
5·1
5·2
5·3
5·4
5·5


123
400,000
5·6


·0
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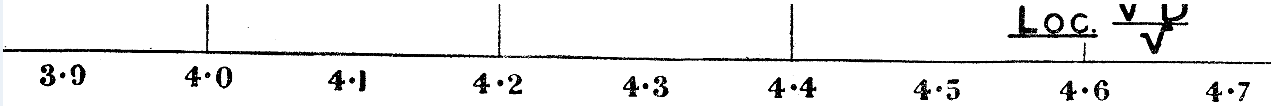

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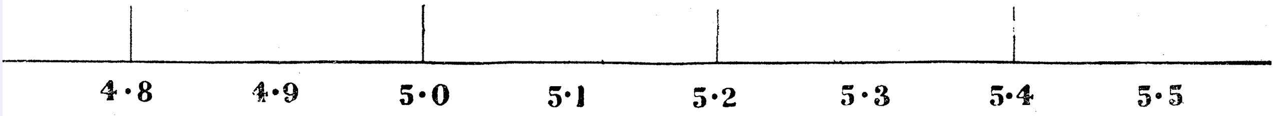

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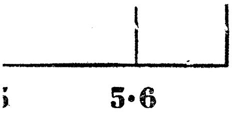

To face p. 206.

EXPERIMENTS on Surface Friction.

Diameter of pipe in centimetres.	Length on which the friction was measured.	Fluid used.	Method of producing flow.	Method of measuring discharge.	Method of measuring friction.
1·255	52·96	Water	(1, 2, 3) Centrifugal pump (4) Plunger pumps	Meter	(1) Mercury tilting gauge. (2) Water U-tube. (3) Mercury U-tube. (4) Bourdon pressure gauges.
0·7125	30·50	”	(1, 2, 3) Centrifugal pump (4) Plunger pumps	”	(1) Mercury tilting gauge. (2) Water U-tube. (3) Mercury U-tube. (4) Bourdon pressure gauges.
0·361	22·86	”	Centrifugal pump Plunger pumps	”	Mercury U-tube. Bourdon pressure gauges.
2·855	61·2	”	Centrifugal pump	”	Mercury tilting gauge.
1·255	52·96	Air	Suction of gas holder Centrifugal fan	Motion of gas holder Pitot tube in axis	Water tilting gauge.
0·7125	30·50	”	Suction of gas holder	Motion of gas holder	” ” ”
0·361	22·86	”	Suction of gas holder	” ” ”	” ” ” Mercury tilting gauge.
2·855	61·2	”	Suction of gas holder Centrifugal fan	” ” ” Pitot tube in axis	Water tilting gauge.
10·10	152·5	”	” ”	” ” ”	” ” ”
10·10	152·5	Thick oil	Centrifugal pump	By direct weighing	Mercury tilting gauge.

Coming to the chief features of interest in the curves of $R/\rho v^2$ it will be noticed that the inclination to the horizontal becomes gradually less as the value of vd/ν increases, indicating that the law of resistance tends to become one in which the friction varies as the square of the velocity and therefore independent of the dimensions and temperature of the fluid. At what value of vd/ν this becomes approximately the case the data available are not sufficient to predict.

As the value of vd/ν diminishes the corresponding values of $R/\rho v^2$ grow at an increasing rate, obtaining a maximum, for pipes of the smoothness used in these experiments, of 0·0054 when vd/ν is equal to 2500 approximately. Throughout the whole of this range in vd/ν of from 2500 to 470,000, with the exception of a few individual determinations due possibly to errors of observation, the variation of $R/\rho v^2$

for either fluid, in any of the four pipes, from its mean value does not exceed 2·0 per cent., so that the similarity of the motions over this range is fully demonstrated. Below $vd/\nu = 2500$ there is a sudden fall in the value of the ordinate showing that a change in the character of the motion is taking place—a change which is revealed to the observer by the rapid fluctuations which take place in the resistance and render accurate determinations practically impossible. It appears that the fluctuations are due to the steepness of the curve in this region, so that small fluctuations of speeds of flow which would produce no appreciable effect on the resistance when the motion was fully eddying, cause such a relatively large change in the resistance that steady readings cannot be obtained.

When this fall in the value of the ordinate occurs, it will be seen that a relatively small reduction in the value of the abscissæ brings all the plotted points on the theoretical curve for stream-line resistance. As the theory of this type of motion is well known and its results have been constantly checked by observations of the resistance, it has not been considered necessary in the case of the air and water experiments to extend the curve further back than to values corresponding to $vd/\nu = 1250$.

It will be noticed from the results plotted in fig. 3 that the values of vd/ν at which the motion changes from sinuous to stream-line in character is practically constant for all the pipes except the smallest (0·361 cm. diameter). In the case of this small pipe, and to a much smaller but perceptible degree in the 0·7125 and 1·255 cm. pipes, it was found that, both for air and water, under the conditions of admission to the pipe, the stream-line motion tended to persist when the critical (2500) value of vd/ν had been exceeded, until at some value of vd/ν depending on the nature of the orifice and the amount of disturbance of the air at the inlet and possibly other factors the value of $R/\rho v^2$ suddenly rose to the value attained in the other pipes under similar conditions.

In explanation of the apparently anomalous behaviour of this small pipe it may be recalled that OSBORNE REYNOLDS defined two critical velocities in pipes. One is the velocity at which a fluid, which enters a pipe in a high state of turbulence, passes from eddying motion to stream-line motion, and which is well defined in all the pipes used in these experiments except the small one under discussion. The other refers to the case in which a perfectly undisturbed fluid enters a pipe in stream-line motion which persists until, with considerable care, a velocity of about seven times that of the former critical velocity can be reached before it breaks down into eddying motion. If slight disturbances are present this breakdown occurs earlier.

It is evident that the behaviour of the small pipe in these experiments is analogous to the second case of critical velocity mentioned by REYNOLDS, and that the turbulence existing outside the orifice to this pipe was not sufficiently violent to correspond to turbulent flow, whereas in the case of the other pipes it was so. Further confirmation of this was found in the fact that by fitting a bell mouth-piece inlet to the pipe to ensure steadier conditions, and making a series of observations of surface friction, the velocity at which the flow became fully eddying, as shown by the ordinates of the curve reaching the same value as for the other pipes, was almost precisely that found by REYNOLDS for his upper limit, *i.e.*, at a value of $vd/\nu = 16000$.

When the mouth-piece was removed and the fluid, which in this case was air, was drawn into the open end of the pipe from the room, by the suction of a gas holder, the observations plotted in fig. 3 were obtained. These show a critical value of vd/ν of about 3000. Beyond this value the results are in complete agreement with the other pipes. Further work in this direction is being made the subject of a separate investigation.

In the experiments on the thick oils it was found impossible, with the pumping appliances available, to reach the critical value of the velocity, so that the experiments were all confined to cases of stream-line motion. The particular oil on which these observations were made had a value of the kinematical viscosity at 15·5° C. of 36·2, or 3230 times that of water at the same temperature. By heating this to 50° C. the kinematical viscosity could be reduced to 2·1, but even at this temperature the critical velocity in the 10 cm. pipe used for the experiments would have been 525 cm. per second. As a matter of interest the results of a series of observations of the surface friction of this oil, when flowing through a steel pipe 10·1 cm. diameter at speeds varying from 5 to 60 cm. per second, are given in Table IV. and are also plotted in fig. 3. As the value of $R/\rho v^2$ in these observations is in some cases as much as 400 times that of its maximum value in the air and water experiments, a separate diagram has been made in the right-hand corner of fig. 3. In this diagram the curve is drawn from the equation $\frac{R}{\rho v^2}\frac{vd}{\nu} = 8$. A table of viscosities determined by a viscosimeter has been used in the calculation of the values of vd/ν for the abscissæ of the plotted points. Since, in all probability, the apparatus used for the experiments constituted a more refined method of determining the viscosity than the viscosimeter itself, any systematic variation of the plotted points from the curve would not necessarily indicate errors of observation, but it will be seen that the agreement is remarkably good.

The Limits of the Index Law of Resistance.

The determination, in these experiments, of the frictional resistance of a 1·255 cm. pipe, when the velocity has ranged from the first commencement of eddying motion, at a speed of flow of 22 cm. per second, up to a speed of 3150 cm. per second, has made possible a check, within these limits, of the accuracy of the well-known index law. This law appears to be due mainly to the observations of WILLIAM FROUDE and OSBORNE REYNOLDS, and there is no doubt that over a moderate range of speed it holds with considerable accuracy. It may be mentioned that the ranges of speeds obtained by OSBORNE REYNOLDS in his experiments on 0·62 and 1·27 cm. pipes were not large, viz., from 48·0 (the critical velocity) to 469 cm. per second in the 0·62 cm. pipe, and from 23 (the critical velocity) to 705 cm. per second in the 1·27 cm. pipe, and further, that the difficulties in maintaining a uniform speed in these experiments, owing to the fact that the water was drawn direct from the Manchester mains, were such as to render any small variation from the index law, such as would exist over such ranges of speeds, almost impossible to detect. In FROUDE's experiments* on towing flat boards the range in speed was not greater than from 50 to 500 cm. per second, so that no deviation would be apparent.

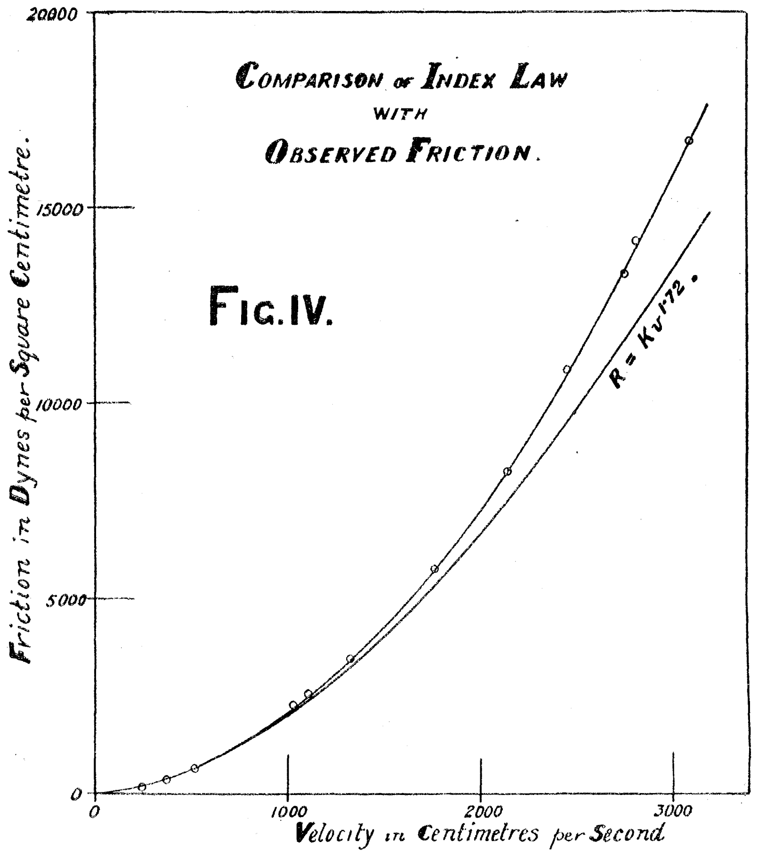

FIG. IV.

Flow of Water in Pipe 1·255 Centimetres Diameter.

In the case of the present experiments the results for the 1·255 cm. pipe were taken and according to REYNOLDS's method, the logarithms of the friction and velocity from $v = 40$ to $v = 100$ cm. per second were carefully plotted. The points so obtained were found to lie on a straight line whose slope was 1·72 to 1. Assuming a law of resistance $R = kv^n$, where n had this value, k was determined from the low speed observations used for the determination of n and a series of values of R were calculated up to a speed of 3200 cm. per second. Plotting these values and those actually obtained in the experiments the two curves in fig. 4 were obtained, from which it will be seen that by the use of the index law the resistance is underestimated by 5 per cent. at

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* 'B.A. Report,' 1872, p. 118.

1000 cm. per second, by 8·5 per cent. at 2000 cm. per second, and by 15 per cent. at 3000 cm. per second.

This, of course, is an extreme case, since if the intermediate speed observations had been taken for the determination of n, a higher value would have been obtained and the error in the use of the index law at the highest velocities would have been much less. In order to show the manner of variation of n throughout the whole range of velocities obtained, the values of n have been determined by the Reynolds method at four different stages and are as follows :—

Velocity in centimetres per second . . .	58.	258.	900.	2250.
Value of n from plotting	1·72	1·77	1·82	1·92.

Similar results showing a gradual increase in the value of n as the velocity increases have been obtained by the reduction of the observations for the 0·7125 and 1·255 cm. pipes, and it may therefore be taken as fully demonstrated that an index law for surface friction cannot be devised which will express the facts with any accuracy, except over a comparatively small range in the value of vd/ν. It will be obvious that this factor must be borne in mind in predicting the skin friction of large bodies moving in a fluid from observations on small-scale models moved in the same fluid.

The Comparison of the Results with those of Previous Experimenters.

As the method of representing the results of the surface friction experiments described in this paper is somewhat novel, it has been considered advisable to reduce the observations of some well-known previous experimenters, in a similar manner, and to plot them all on the same diagram for the purpose of comparison. Taking only experiments on pipes whose roughness was comparable with the ones used by the authors, those chosen for reduction are as follows :—

Experimenter.		Fluid used.	Nature of surface.	Diameter of pipes in centimetres.
DARCY	'Comptes Rendus de l'Académie des Sciences,' vol. 38	Water	Drawn lead	2·7 and 4·1
"		"	Bitumen covered	8·26, 19·6, and 28·5
REYNOLDS	'Phil. Trans. Roy. Soc.,' 1883	"	Drawn lead	0·62 and 1·27
SAPH and SCHODER	'Proc. Amer. Soc. Civ. Engs.,' 1903, vol. 51, p. 253	"	Drawn brass	5·31, 3·81, 3·14, 2·68, 1·60, 0·82, and 0·27
BRIX	'Phil. Mag.,' vol. 17, 1909, p. 395	Air	Lead	0·635
"		"	Wrought iron	8·26, 13·2, and 17·1
STOCKALPER	'Revue Univers. des Mines,' vol. 7, p. 257	"	Cast and wrought iron	15 and 20

Unfortunately in some of these observations the temperature has not been recorded so that the values of vd/ν calculated are only approximate. The reduced results have been plotted in fig. 5 where, for convenience of comparison, the limits of the results of the present experiments have been indicated by broken lines. In the case of DARCY's results the points referring to the two lead pipes and the bitumen-covered pipes of 8·6 and 19·6 cm. diameter have been taken as forming one group with approximately the same surface smoothness and a line has been drawn through their mean position. This line shows the same characteristics as the curve of the present experiments, except that the surface is somewhat rougher. The bitumen-covered pipe of 28·5 cm. diameter had evidently a considerably rougher surface, but otherwise the distribution of the points are what would be expected. For REYNOLDS's experiments on the two lead pipes a mean line has also been drawn which indicates a lower resistance than that found in the pipes used for the present experiments, but the character of the curve and the position of the critical point are identical with those now found. From the point of view of comparison with the present observations neither of the above sets of experiments are so satisfactory as those of SAPH and SCHODER,[*] since the nature of the surfaces in the two researches was probably identical. The plottings of these, with a very few exceptions, lie within the limits of the present experiments of which they form an excellent check. The whole series of these experiments include observations on pipes of 5·32, 3·81, 3·14, 2·68, 2·08, 1·60, 1·27, 0·95, 0·82, 0·72, 0·66, 0·56, 0·46, 0·38 and 0·272 diameter, of which only the 5·32, 3·81, 3·14, 2·68, 1·60, 0·82 and 0·272 cm. have been reduced for plotting in fig. 5. For the determination of the frictional resistance in the case of the flow of air the amount of reliable data appears to be very small. The six experiments of Dr. BRIX[†] on iron pipes of 8·26, 13·2, and 17·1 cm. diameter are in fair agreement with the present results, but the interpretation of the series of observations on the 0·635 cm. lead pipe is difficult as the form of the curve appears to indicate that the conditions are those of stream-line flow. The only explanation which can be suggested is that the diameter of the pipe was over-estimated by some 15 per cent., in which case the curve shown through the plotted points would come into fair agreement with the theoretical curve for stream-line flow. The results of STOCKALPER's experiments, which possess additional interest owing to the fact that the air was under a pressure of about 5 atmospheres, are well in agreement with the present experiments as regards the 20 cm. pipe and less so in the case of the 15 cm. pipe, but unfortunately the range of velocity is not a large one.

In conclusion, a note on the various deductions from theoretical considerations of the criterion for similarity of motion in fluids may be of interest. In a paper read before the Cambridge Philosophical Society in 1850, STOKES[‡] considered any number of similar systems composed of similar solids, oscillating in a similar manner in different

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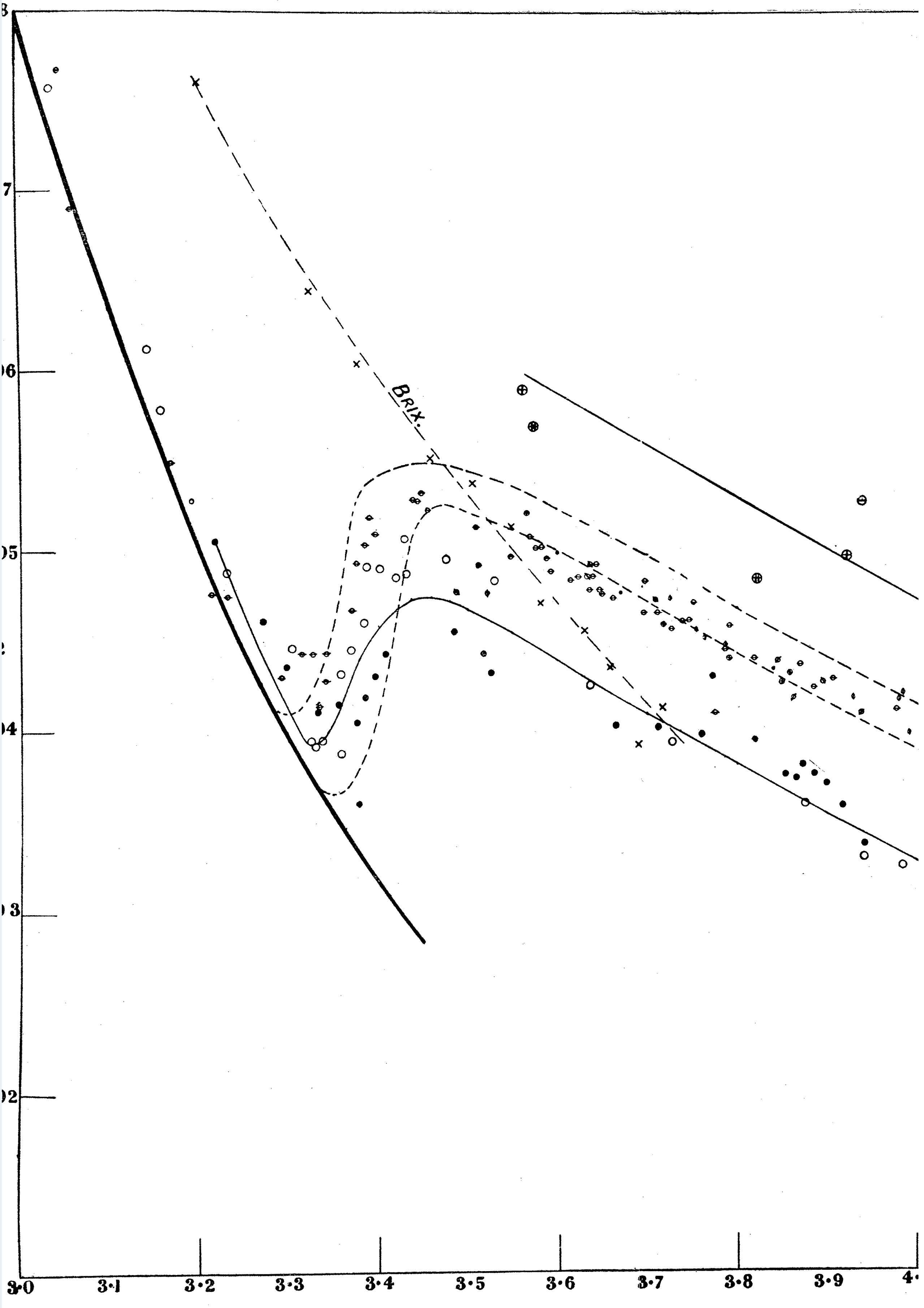
[*] 'Proc. Amer. Soc. Civil Engs.,' 1903, vol. 51, p. 253.

[†] 'Phil. Mag.,' vol. 17, 1909, p. 395.

[‡] STOKES, 'Mathematical and Physical Papers,' vol. 3, p. 17.


BRIX.
7
6
5
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3
2
3·0
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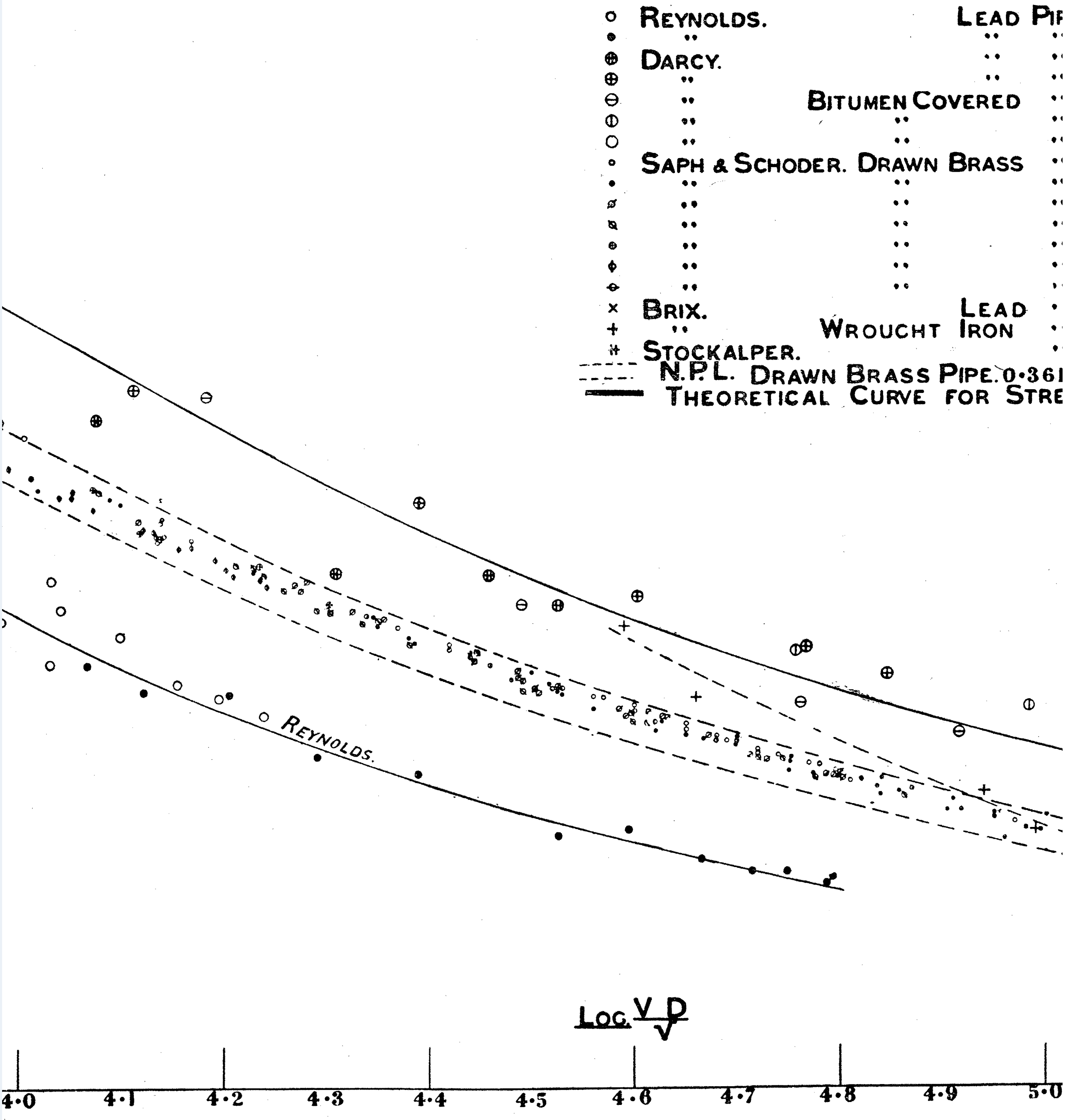
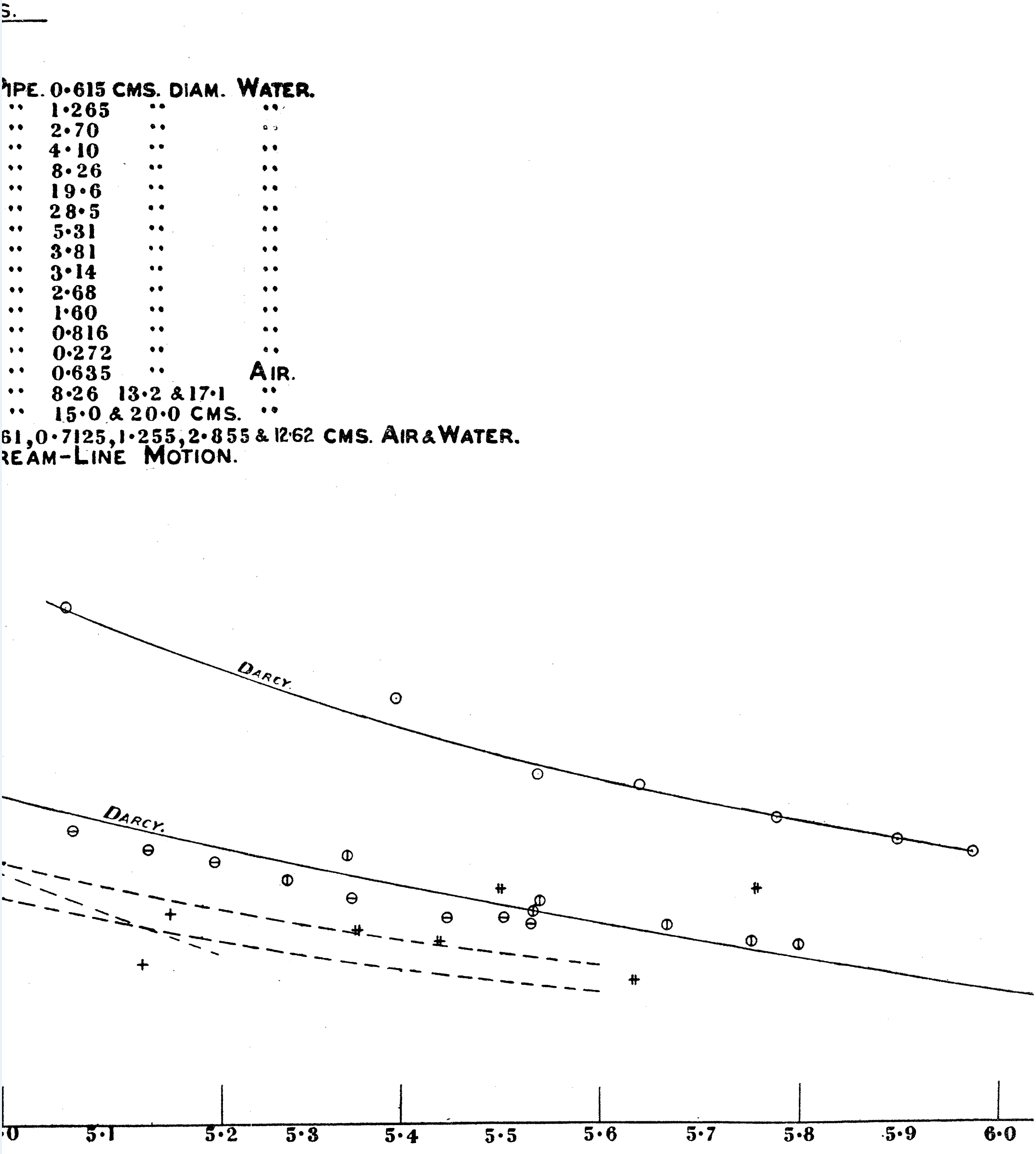
3·7
3·8
3·9
4·

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

FIG. V.

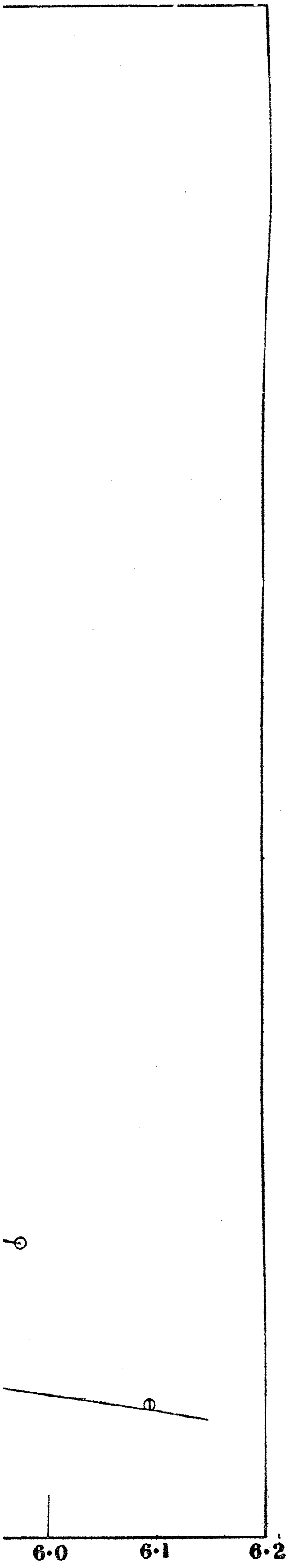
COMPARISON OF RESULTS WITH THOSE OF PREVIOUS EXPERIMENTERS.




S.
IPE. 0·615 CMS. DIAM. WATER.
.. 1·265 .. ..
.. 2·70 .. ..
.. 4·10 .. ..
.. 8·26 .. ..
.. 19·6 .. ..
.. 28·5 .. ..
.. 5·31 .. ..
.. 3·81 .. ..
.. 3·14 .. ..
.. 2·68 .. ..
.. 1·60 .. ..
.. 0·816 .. ..
.. 0·272 .. ..
.. 0·635 .. AIR.
.. 8·26 13·2 & 17·1 ..
.. 15·0 & 20·0 CMS. ..
61, 0·7125, 1·255, 2·855 & 12·62 CMS. AIR & WATER.
REAM-LINE MOTION.
DARCY.
DARCY.
·0
5·1
5·2
5·3
5·4
5·5
5·6
5·7
5·8
5·9
6·0

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Trans., *A*, *vol.* 214.



To face p. 212.

fluids or in the same fluid. Then, if a, a^1, ..., be homologous lines in the different systems, T, T^1, ..., corresponding times, such as the times of oscillation, c, c^1, ..., the maximum excursions of similarly situated points in the fluids and the equations

$$\frac{dp}{dx} = \mu \left\{\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right\} - \rho\,\frac{du}{dt},$$

&c., &c.,

(which are identical with the general equations of motion of a viscous fluid, in which the terms involving the squares of the velocities are neglected) are satisfied for one system, they will be satisfied for all the systems provided

$$u \propto v \propto w, \qquad x \propto y \propto z,$$

and

$$p \propto \frac{\mu u}{x} \propto \frac{\rho u x}{t},$$

i.e., provided

$$u \propto \frac{c}{\mathrm{T}}, \qquad x \propto a, \qquad t \propto \mathrm{T}, \qquad \text{and} \qquad \frac{a^2}{\mathrm{T}} \propto \frac{\mu}{\rho}.$$

From this last relation it follows that for dynamical similarity the value of ua/ν must be constant for all the systems.

In 1873, HELMHOLTZ,* in a paper to the Royal Prussian Academy of Sciences, Berlin, gave a somewhat more general treatment of the question. Considering two fluids of densities ρ_1, ρ_2, and kinematical viscosities ν_1, ν_2, the conditions under which the motions of the two fluids are similar are determined thus :—

Taking the equation of motion of the first fluid as

$$-\frac{1}{\rho_1}\frac{dp}{dx} = \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz} - \nu_1\left\{\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right\},$$

and two similar equations, and writing

$$\nu_2 = q\nu_1, \qquad \rho_2 = r\rho_1,$$

then in order that these equations may be transformed into the equations for the second fluid

$$-\frac{1}{\rho_2}\frac{d\mathrm{P}}{d\mathrm{X}} = \frac{d\mathrm{U}}{d\mathrm{T}} + \mathrm{U}\frac{d\mathrm{U}}{d\mathrm{X}} + \mathrm{V}\frac{d\mathrm{U}}{d\mathrm{Y}} + \mathrm{W}\frac{d\mathrm{U}}{d\mathrm{Z}} - \nu_2\left\{\frac{d^2\mathrm{U}}{d\mathrm{X}^2} + \frac{d^2\mathrm{U}}{d\mathrm{Y}^2} + \frac{d^2\mathrm{U}}{d\mathrm{Z}^2}\right\},$$

&c., under the given conditions of similarity

$$\mathrm{U} = nu, \qquad \mathrm{V} = nv, \qquad \mathrm{W} = nw,$$

---

* HELMHOLTZ, 'Wissenschaftliche Abhandlungen,' vol. I., p. 158.

it will be seen that, multiplying (1) by $\frac{n^3}{q}$ the required transformation is effected if the change in the scales of length, time and pressure are

$$X = \frac{q}{n}x, \qquad T = \frac{q}{n^2}t, \qquad P = n^2 rp.$$

It follows, therefore, from the linear scale relation that, if l and L are corresponding linear dimensions of two pipes in which the fluids of densities ρ_1, ρ_2, and kinematical viscosities ν_1, ν_2 are flowing, in order that the two motions may be similar $L = \frac{\nu_2 v}{\nu_1 V} l$, or $\frac{VL}{\nu}$ must have the same value for each. Again, from the pressure scale relation it follows that for similar motion the value of $P/\rho V^2$ is the same for each fluid.

The method of Lord RAYLEIGH, which was first applied in considering the size of drops formed under various conditions,* is as follows. Assuming that the resistance depends solely on the velocity, linear dimensions, viscosity and density, and also that the resistance F varies as $\rho^a L^b v^c \mu^d$ then if M, L, T are the units of mass, length, and time, the dimensions of F are

$$\left(\frac{M}{L^3}\right)^a L^b \left(\frac{L}{T}\right)^c \left(\frac{M}{LT}\right)^d = \frac{ML}{T^2},$$

so that

$$a = 1-d, \qquad b = 2-d, \qquad c = 2-d,$$

and therefore

$$F \propto \rho v^2 L^2 \left(\frac{\mu}{\rho v L}\right)^d,$$

and the resistance per unit area can be written

$$R = \rho v^2 f(vL/\nu).$$

With reference to NEWTON'S Theorem on Similar Motions, in Proposition 32, Book II., of the 'Principia,' the authors are indebted to Dr. R. T. GLAZEBROOK for the following note:—

In this Theorem NEWTON shows that two systems of particles, if started similarly, will continue to move in a similar manner if the acceleration of each system is proportional to V^2/L, V being the velocity and L a linear quantity defining the dimensions of the system—the diameter of a particle. Now denoting by f the acceleration and by ρ the density, we have for fluid friction

$$\frac{L^3 \rho f}{L^2} = \text{force per unit area} = \mu \frac{dV}{dx}\cdot$$

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* 'Phil. Mag.,' 1899, vol. 48, p. 321.

Thus the dimensional equation is

$$[f] = \left[\frac{\mu}{\rho}\right]\left[\frac{\mathrm{V}}{\mathrm{L}^2}\right]$$

But according to Newton for similar motion

$$[f] = \left[\frac{\mathrm{V}^2}{\mathrm{L}}\right].$$

Hence

$$\left[\frac{\mu}{\rho}\right]\left[\frac{\mathrm{V}}{\mathrm{L}^2}\right] = \left[\frac{\mathrm{V}^2}{\mathrm{L}}\right],$$

or $\frac{\mu}{\rho}\frac{1}{\mathrm{L}\,.\,\mathrm{V}}$ is independent of dimensions.

The authors also desire to thank Dr. Glazebrook for the facilities given them for carrying out the work and the interest he has shown in the progress of the experiments.

Table I.—Observations of Mean and Maximum Velocities.

Mean velocity, centimetres per second.	Observed velocity at axis (maximum).	Rate of mean velocity to maximum velocity.	Nature of flow.
167·5	208·8	0·803	Water in pipe, 2·855 cm. diameter.
142·3	177·5	0·802	
129·9	162·0	0·802	
121·1	150·9	0·802	
105·7	132·4	0·798	
94·6	118·4	0·799	
94·5	118·6	0·798	
85·7	107·4	0·798	
75·7	95·0	0·797	
62·8	79·0	0·795	
42·4	53·7	0·789	
37·0	47·1	0·786	
153·4	191·4	0·802	
145·9	181·2	0·805	
160·6	200·0	0·803	
167·5	207·8	0·806	
217·2	269·4	0·807	
187·5	232·8	0·806	
194·6	246·7	0·789	Water in pipe, 0·7125 cm. diameter.
145·9	187·0	0·780	
125·9	161·9	0·778	
104·4	135·5	0·770	
81·73	107·0	0·764	
65·20	87·07	0·748	

TABLE I.—Observations of Mean and Maximum Velocities (continued).

Mean velocity, centimetres per second.	Observed velocity at axis (maximum.)	Rate of mean velocity to maximum velocity.	Nature of flow.
57·60	77·83	0·740	Water in pipe, 0·7125 cm. diameter.
49·06	67·91	0·722	
39·44	65·70	0·619	
35·12	61·37	0·572	
33·15	58·79	0·564	
28·89	55·60	0·519	
27·28	53·69	0·508	
25·24	49·05	0·515	
24·24	48·63	0·498	
22·25	43·28	0·514	
512·9	642·4	0·799	
450·0	563·3	0·799	
374·8	469·6	0·799	
282·3	356·1	0·793	
668·1	828·6	0·805	
733·0	906·8	0·808	
797·9	991·1	0·805	
872·0	1076	0·810	
940·4	1165	0·807	
1028	1264	0·813	
1117	1376	0·812	
1199	1474	0·814	
573·9	757·2	0·758	Air in pipe, 1·255 cm. diameter.
720·3	937·6	0·768	
115·5	149·0	0·775	
648·8	853·0	0·761	
471·5	631·3	0·747	
363·6	515·4	0·705	
404·2	549·7	0·735	
410·8	558·9	0·735	
893·6	1161	0·770	
240·7	317·4	0·758	Air in pipe, 2·855 cm. diameter.
283·6	372·3	0·762	
207·7	281·8	0·745	
504·5	840·0	0·611	Air in pipe, 0·7125 cm. diameter.
290·5	582·2	0·499	
333·8	669·5	0·498	
364·0	712·2	0·511	
389·0	766·0	0·508	
256·2	505·7	0·505	
324·0	646·2	0·501	

TABLE II.—Experiments with Water.

Mean velocity, centimetres per second. (v.)	Surface friction, dynes per square centimetre. (R.)	Value of $\frac{R}{\rho v^2} 10^2$.	Value of $\frac{vd}{\nu} 10^{-3}$.	Temperature.
				° C.
Pipe No. 1. Diameter 2·855 cm.				
116·3	41·8	0·309	25·32	10·2
125·6	47·8	0·303	27·36	
138·2	56·2	0·294	30·10	
149·7	64·6	0·295	32·62	
136·8	55·2	0·294	30·00	
155·8	68·5	0·282	34·20	10·5
180·0	88·0	0·272	39·5	
201·0	109·2	0·270	44·1	10·7
214·2	120·2	0·261	47·4	
95·0	29·46	0·327	21·0	
104·1	34·20	0·316	23·0	11·1
95·0	29·52	0·328	21·0	
84·8	24·4	0·339	18·8	
73·6	18·95	0·350	16·5	
63·0	14·44	0·363	14·1	
53·7	10·93	0·380	12·0	
46·0	8·26	0·390	10·3	11·3
42·7	7·22	0·396	9·60	
38·2	5·95	0·408	8·58	
33·1	4·74	0·433	7·44	
27·0	3·27	0·449	6·12	
22·3	23·9	0·481	5·06	
29·6	3·86	0·440	6·70	11·5
Pipe No. 16. Diameter 1·255 cm.				
258	199	0·299	29·18	16·0
299	259	0·288	33·80	
343	327	0·278	38·8	
344	331	0·280	38·9	
343	328	0·278	38·8	
258	198	0·298	29·2	
501	640	0·255	56·6	15·8
461	556	0·262	52·2	
423	479	0·268	47·8	
386	404	0·271	43·7	
299	238	0·288	33·8	
447	526	0·263	50·6	
490	612	0·255	55·4	
528	689	0·247	59·7	
552	762	0·250	62·4	
584	842	0·247	66·0	
612	918	0·245	69·2	
632	965	0·242	71·4	
673	1078	0·238	76·2	
699	1162	0·238	79·0	

TABLE II.—Experiments with Water (continued).

Mean velocity, centimetres per second. (v.)	Surface friction, dynes per square centimetre. (R.)	Value of $\frac{R}{\rho v^2} 10^2$.	Value of $\frac{vd}{\nu} 10^{-3}$.	Temperature.
				° C.
Pipe No. 16. Diameter 1·255 cm. (continued).				
730	1247	0·234	82·5	15·8
730	1252	0·235	82·5	
757	1337	0·230	85·6	
796	1460	0·231	90·0	
524	693	0·252	59·2	
232	162	0·301	26·2	
243	177	0·300	27·5	
197	124	0·320	22·3	
151	78	0·342	17·1	
183	108	0·323	20·7	
226	157	0·308	24·6	14·5
199	124·2	0·314	21·7	14·3
201	128·4	0·318	21·9	
193	120·4	0·323	21·0	
187	114·5	0·327	20·4	
173	99·5	0·332	18·9	
163	89·4	0·337	17·8	
152	80·4	0·348	16·6	
136	65·9	0·356	14·9	
170	96·6	0·334	18·7	15·0
169	86·8	0·339	17·6	
147	75·1	0·348	16·2	
134	64·3	0·358	14·8	
116	51·1	0·380	12·8	
116	50·3	0·374	12·8	
129	59·2	0·356	14·2	
124	54·5	0·355	13·7	
115	49·1	0·371	12·7	
105·5	42·4	0·381	11·6	
95·6	36·3	0·397	10·5	
88·4	30·8	0·388	9·75	
79·6	25·6	0·404	8·78	
69·0	19·7	0·413	7·60	
58·1	15·0	0·444	6·40	
54·4	13·5	0·456	6·00	
49·6	11·4	0·461	5·48	
42·7	8·8	0·482	4·70	
36·4	6·4	0·481	4·00	
30·5	4·9	0·526	3·36	
54·0	13·1	0·449	6·12	16·0
61·6	16·6	0·437	6·95	
64·2	17·7	0·430	7·28	
61·7	16·5	0·433	6·98	
33·1	5·6	0·514	3·74	
38·6	7·3	0·488	4·36	
28·3	4·2	0·530	3·20	
26·1	3·6	0·529	2·96	
23·1	2·8	0·521	2·61	
19·6	1·8	0·472	2·22	

TABLE II.—Experiments with Water (continued).

Mean velocity, centimetres per second. (v.)	Surface friction, dynes per square centimetre. (R.)	Value of $\frac{\mathrm{R}}{\rho v^2} 10^2$.	Value of $\frac{vd}{\nu} 10^{-3}$.	Temperature.
				° C.
Pipe No. 16. Diameter 1·255 cm. (continued).				
705	1172	0·235	77·0	16·0
798	1480	0·232	87·2	
884	1776	0·227	96·6	
964	2075	0·223	105·2	
1100	2650	0·219	120·2	
1036	2363	0·220	113·2	
38·7	7·28	0·486	4·37	14·5
26·3	3·71	0·536	2·97	
23·5	2·90	0·526	2·66	
19·5	1·63	0·429	2·20	
27·3	3·94	0·530	3·09	
24·8	3·29	0·536	2·80	
140·0	68·8	0·351	15·80	
171·0	96·9	0·331	19·35	
1322	3440	0·197	178	23·0
1767	5755	0·1845	238	
2150	8322	0·1813	289	
2446	10860	0·1816	330	
2762	13360	0·1748	373	
3017	16320	0·1795	407	
1322	3407	0·1950	178	
1800	5796	0·1788	243	
2813	14170	0·1790	380	
3090	16700	0·1750	418	
3188	17660	0·1738	430	
Pipe No. 17. Diameter 0·7125 cm.				
245	225	0·375	13·00	9·3
193	145·9	0·392	10·26	
170·5	117·2	0·403	9·06	
144·4	89·4	0·430	7·74	9·6
114·2	59·6	0·457	6·12	
281	278	0·353	15·27	10·1
316	347	0·348	17·17	
496	742	0·302	27·2	10·4
427	576	0·316	23·4	
356	427	0·337	19·5	
574	962	0·292	31·5	
640	1156	0·282	35·1	
718	1433	0·278	37·6	8·6
822	1847	0·273	43·1	
920	2230	0·264	48·2	
1004	2620	0·260	52·6	

TABLE II.—Experiments with Water (continued).

Mean velocity, centimetres per second. (v.)	Surface friction, dynes per square centimetre. (R.)	Value of $\frac{R}{\rho v^2} 10^2$.	Value of $\frac{vd}{\nu} 10^{-3}$.	Temperature.
				° C.
Pipe No. 17. Diameter 0·7125 cm. (continued).				
1094	3025	0·253	57·7	8·9
1282	4060	0·247	67·6	
1186	3490	0·248	62·5	9·0
169·1	116·0	0·406	9·13	
139·0	84·0	0·435	7·51	
77·5	29·9	0·498	4·18	9·7
68·3	23·8	0·508	3·68	
58·2	17·9	0·528	3·14	
101·3	47·7	0·465	5·47	10·6
49·3	13·0	0·535	2·66	9·8
46·8	11·8	0·540	2·53	
49·4	13·1	0·536	2·66	
44·9	10·8	0·538	2·42	
44·5	10·2	0·518	2·40	
95·5	43·2	0·473	5·15	
110·6	55·8	0·456	5·96	
5254	47500	0·173	415	25·0
5200	45950	0·170	412	
5260	47700	0·1725	416	
2987	16750	0·188	236	
3840	25930	0·176	304	
4180	30700	0·176	331	
1670	58700	0·210	132	
4010	28070	0·175	317	
5070	44350	0·172	402	
4980	42100	0·170	394	
4355	32930	0·174	344	
4470	35350	0·177	354	
4630	36950	0·172	367	
1978	7950	0·203	156	
1117	2970	0·234	88·5	
1353	3970	0·221	107·1	
1485	4665	0·211	117·5	
1776	6490	0·206	140·6	
Pipe No. 18. Diameter 0·361 cm.				
5595	61850	0·1975	202·0	20·5
5410	58330	0·1990	195·3	
5062	52650	0·2045	182·8	
4845	47970	0·2040	175·0	
4425	41420	0·2110	159·7	
3865	31410	0·2100	139·2	
3673	29420	0·2180	132·6	
3140	21880	0·2220	113·3	

TABLE II.—Experiments with Water (continued).

Mean velocity, centimetres per second. (v.)	Surface friction, dynes per square centimetre. (R.)	Value of $\frac{R}{\rho v^2}10^2$.	Value of $\frac{vd}{\nu}10^{-3}$.	Temperature.
				° C.
Pipe No. 18. Diameter 0·361 cm. (continued).				
2940	19280	0·2235	106·2	20·5
4640	45000	0·2090	167·6	
4920	49800	0·2050	177·4	
4625	44350	0·2070	167·0	
4325	39000	0·2080	156·2	
3970	33500	0·2120	143·2	
5606	61850	0·1972	202·5	
596	1226	0·346	17·7	12·0
733	1746	0·325	21·2	
873	2353	0·309	25·3	
994	2975	0·302	28·7	
1138	2352	0·290	32·8	
1227	4260	0·283	35·6	
750	1812	0·323	22·0	12·5
602	1233	0·341	17·05	
603	1232	0·339	17·05	

TABLE III.—Experiments with Air.

Mean velocity, centimetres per second. (v.)	Surface friction, dynes per square centimetre. (R.)	Value of $\frac{R}{\rho v^2}10^2$.	Value of $\frac{vd}{\nu}10^{-3}$.
Pipe No. 1. Diameter 2·855 cm. Temperature 15° C.			
232·2	0·335	0·506	4·48
252·6	0·384	0·491	4·87
211·2	0·281	0·515	4·07
195·8	0·242	0·514	3·78
186·4	0·224	0·525	3·59
170·1	0·183	0·517	3·28
178·5	0·206	0·528	3·44
278·8	0·456	0·482	5·37
302·0	0·524	0·468	5·82
328·0	0·607	0·458	6·32
346	0·648	0·437	6·67
274	0·430	0·468	5·29
478	1·103	0·394	9·21
404	0·844	0·422	7·78
488	1·190	0·407	9·41
353	0·664	0·434	6·81

TABLE III.—Experiments with Air (continued).

Mean velocity, centimetres per second. (v.)	Surface friction, dynes per square centimetre. (R.)	Value of $\frac{R}{\rho v^2} 10^2$.	Value of $\frac{vd}{\nu} 10^{-3}$.
Pipe No. 1. Diameter 2·855 cm. Temperature 15° C. (continued).			
311	0·544	0·457	6·00
607	1·703	0·378	11·70
675	2·09	0·374	13·00
755	2·53	0·362	14·56
985	4·02	0·348	19·00
1140	5·20	0·325	22·00
1645	10·00	0·302	31·70
1512	8·48	0·303	29·2
1458	8·02	0·307	28·1
1410	7·56	0·310	27·2
1347	6·98	0·313	26·0
1235	5·98	0·320	23·8
1195	5·65	0·323	23·0
1102	4·87	0·327	21·2
Steel Pipe, 10·1 cm. Diameter.			
279	318	0·333	19·1
Pipe No. 16. Diameter 1·255 cm. Temperature 15° C.			
655	2·492	0·476	5·56
383	0·961	0·534	3·25
350	0·830	0·553	2·97
421	1·155	0·532	3·57
482	1·446	0·509	4·09
803	3·570	0·452	6·81
883	4·203	0·440	7·49
454	1·283	0·509	3·85
570	1·952	0·490	4·84
732	3·010	0·458	6·21
537	1·737	0·491	4·55
404	1·062	0·531	3·43
606	2·218	0·493	5·14
323	0·664	0·519	2·74
440	1·206	0·509	3·73
765	3·290	0·459	6·49
978	5·080	0·434	8·30
1152	6·675	0·411	9·78
1317	8·410	0·397	11·18
1465	10·020	0·381	12·43
1051	5·735	0·423	8·91
984	5·060	0·428	8·35
1530	10·930	0·382	12·98

TABLE III.—Experiments with Air (continued).

Mean velocity, centimetres per second. (v.)	Surface friction, dynes per square centimetre. (R.)	Value of $\frac{R}{\rho v^2} 10^2$.	Value of $\frac{vd}{\nu} 10^{-3}$.
Pipe No. 17. Diameter 0·7125 cm. Temperature 15°·5 C.			
460	0·971	0·374	2·215
313	0·617	0·514	1·51
451	0·931	0·373	2·17
579	2·175	0·530	2·78
784	3·800	0·505	3·377
662	2·850	0·530	3·185
579	2·125	0·519	2·780
576	2·075	0·510	2·770
564	1·920	0·493	2·715
557	1·840	0·484	2·768
552	1·790	0·481	2·655
541	1·600	0·447	2·600
504	1·255	0·403	2·425
417	0·870	0·408	2·010
507	1·365	0·433	2·440
536	1·690	0·478	2·580
557	1·850	0·489	2·680
550	1·860	0·503	2·645
441	0·940	0·395	2·120
497	1·175	0·388	2·390
528	1·527	0·449	2·540
387	0·799	0·436	1·860
374	0·779	0·455	1·800
356	0·754	0·484	1·712
305	0·632	0·554	1·468
347	0·706	0·479	1·670
212	0·423	0·768	1·012
285	0·576	0·582	1·370
300	0·602	0·548	1·440
268	0·548	0·627	1·288
273	0·543	0·598	1·310
358	0·727	0·464	1·722
336	0·679	0·490	1·620
327	0·648	0·496	1·570
300	0·610	0·552	1·445
1047	6·440	0·480	5·030
1152	7·750	0·467	5·535
1224	8·360	0·456	5·890
1245	8·560	0·451	5·995
1384	10·300	0·439	6·655
857	4·460	0·497	4·130
Pipe No. 18. Diameter 0·361 cm. Temperature 16°·5 C.			
849	3·46	0·391	2·06
594	2·45	0·566	1·44
344	1·41	0·970	0·838
274	1·11	1·210	0·667
191	0·786	1·756	0·465

TABLE III.—Experiments with Air (continued).

Mean velocity, centimetres per second. (v.)	Surface friction, dynes per square centimetre. (R.)	Value of $\frac{R}{\rho v^2} 10^2$.	Value of $\frac{vd}{\nu} 10^{-3}$.
Pipe No. 18.	Diameter 0·361 cm.	Temperature 16°·5 C. (continued).	
1080	4·66	0·326	2·63
1270	6·84	0·344	3·10
1310	7·94	0·376	3·19
1310	10·12	0·477	3·20
2390	32·1	0·469	5·95
2710	39·8	0·454	6·80
3040	48·7	0·445	7·67
3370	56·9	0·421	8·55
2020	23·5	0·476	5·01
1580	15·6	0·512	3·91
2050	24·4	0·480	5·25
1370	12·2	0·527	3·39
1740	23·0	0·500	4·30
1260	10·3	0·534	3·10
Pipe No. 12A.	Diameter 12·62 cm.	Temperature 16°·5 C.	
2183	11·90	0·205	185
2692	16·60	0·188	228
3172	22·15	0·180	269
3790	30·65	0·175	321
4025	34·55	0·175	341
1342	4·84	0·220	114
1004	2·92	0·238	85

TABLE IV.—Experiments with Thick Oil.

Diameter of Pipe 10·13 cm.

Mean velocity, centimetres per second.	Temperature, C.	Density.	Value of ρv^2.	Value of ν.	Surface friction, dynes per square centimetre. R.	$\frac{R}{\rho v^2} 10^2$.	Value of $\frac{vd}{\nu}$.
45·9	39·5	0·928	1955	3·79	122·6	6·27	123
10·25	37·7	0·929	975	4·29	30·9	31·7	24
35·9	37·2	0·929	1195	4·41	109·8	9·18	82·5
54·9	36·8	0·930	2800	4·58	177·0	6·33	122·0
8·02	36·4	0·930	59·7	4·76	27·2	45·6	17·0
5·12	35·9	0·930	24·4	4·99	17·5	70·0	10·4
57·4	35·9	0·930	3060	4·99	205·0	6·70	117
30·5	35·6	0·930	865	5·11	110·8	12·81	60·6
5·46	35·9	0·930	27·7	4·99	19·5	70·4	11·1
5·80	34·8	0·931	31·2	5·46	22·8	73·1	10·8
55·80	34·5	0·931	2900	5·57	219·3	7·57	101·5

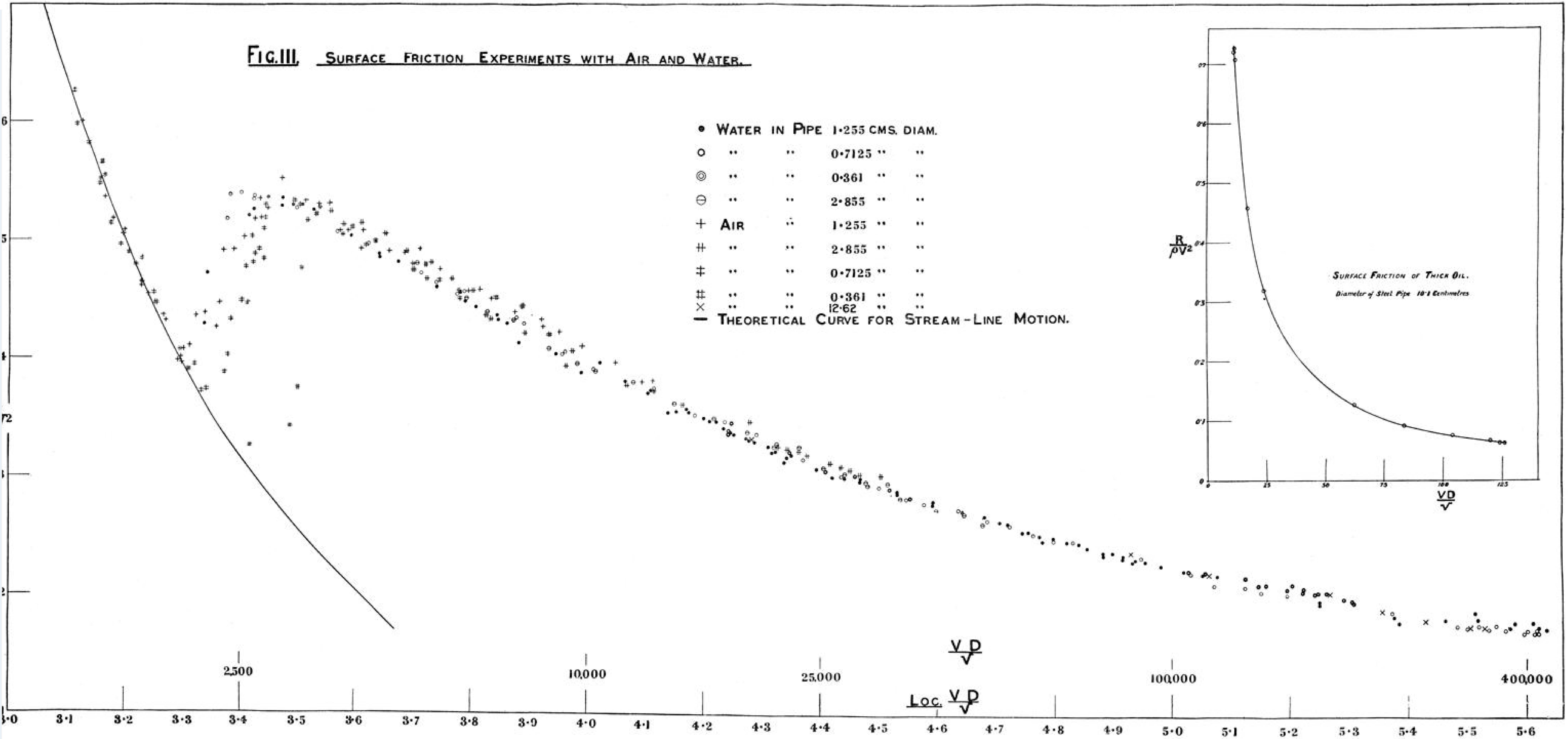

FIG. III. SURFACE FRICTION EXPERIMENTS WITH AIR AND WATER.

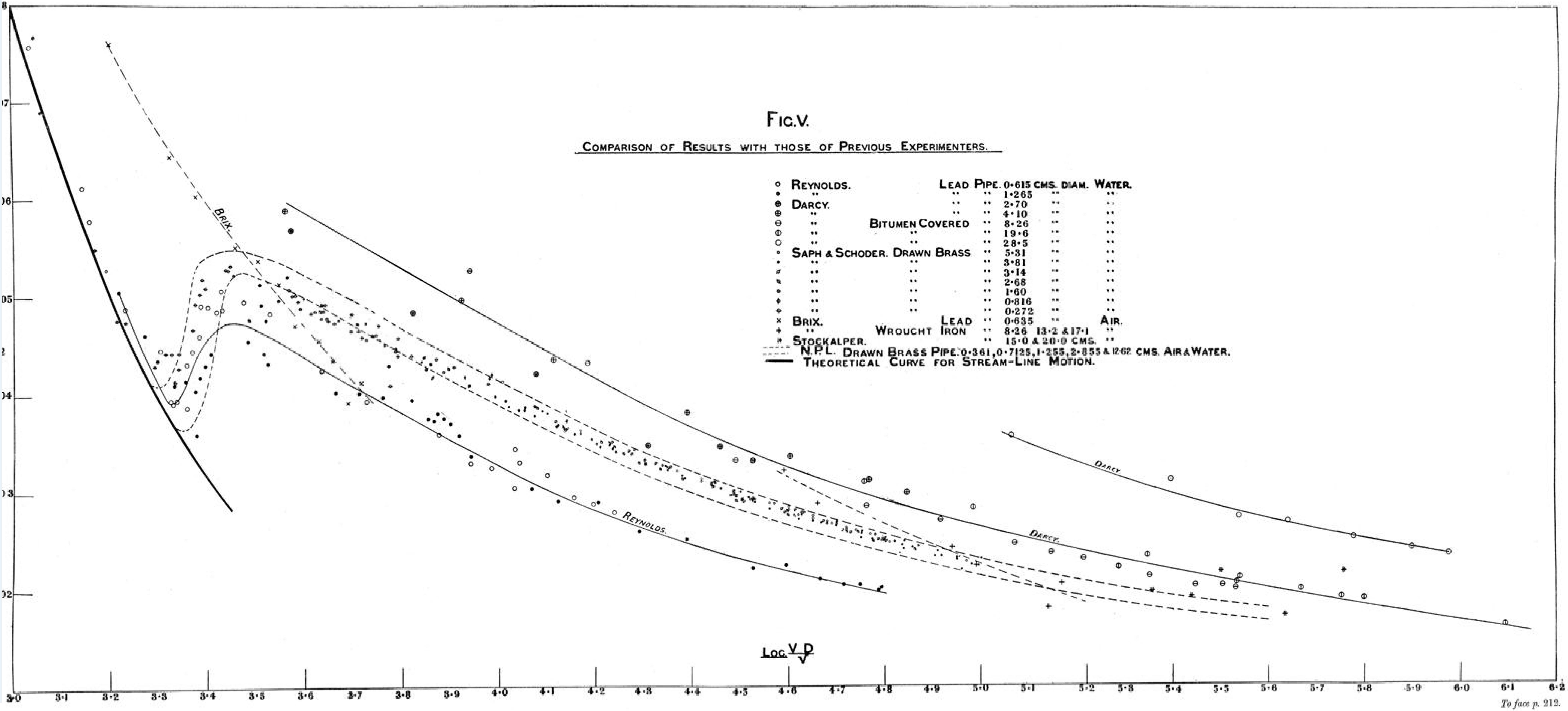

FIG. V.

COMPARISON OF RESULTS WITH THOSE OF PREVIOUS EXPERIMENTERS.